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CONTROLLER DESIGN FOR NONLINEAR AND TIME VARYING PLANTS

by Richard V. Monopoli

Prepared under Grant No. NsG-309-63 *by*
UNIVERSITY OF CONNECTICUT
Storrs, Conn.
for

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CONTROLLER DESIGN FOR NONLINEAR AND TIME VARYING PLANTS

Richard V. Monopoli*

SUMMARY

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A technique for controller design previously reported[2,4], is generalized herein to include a wide class of plant nonlinearities. The technique, based on Liapunov's Second Method, generates a control signal which forces the plant states to be close to the states of a model reference. In addition to the generalization, the technique is extended to include convergence time, i.e. time of response to initial perturbations of the equilibrium, as part of the design problem. Also, modifications to the original technique are made ~~which~~ make it more attractive from an engineering point of view.

Author

INTRODUCTION

Much of the literature dealing with Liapunov's Second Method concerns itself with analysis of stability problems. Notable exceptions, in which the method is applied to engineering design problems, include works by Bass[1], Grayson[2,3], Monopoli[4], Johnson[5], and Nahi[6].

In this report, a method is presented for design of controllers employing a model reference which generalizes the technique due to Grayson. The generalization is to a much broader class of nonlinear plants than previously reported[4]. In addition to the generalization, the technique is extended to include convergence time, i.e. time of response to initial perturbations of the equilibrium, as part of the design problem. Also, certain modifications of Grayson's techniques are made which result in a reduced control signal level and avoid the need for using derivatives of the input signal in generating the control signal.

The technique is applied to plants which include square law damping, static friction, coulomb friction, and hard and soft spring type nonlinearities. Results of digital computer simulations for these plants are presented. These do not exhaust the possibilities, but serve only to demonstrate the method. In fact, the technique is applicable to a wide variety of nonlinear plants provided the form of the nonlinearity is known. In general, the control law requires that the form of the nonlinear function of the plant states be generated. However, if an upper bound on the magnitude of the argument of the nonlinear function is known, this requirement can be relaxed and the exact form of nonlinearity can be replaced by the magnitude of its argument. These points are clarified through the several examples presented.

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The question of convergence time, i.e. the time required for the plant states to become equal to the model states, has not been treated previously in a design context. In the event that the plant is tracking the model, the control action is such that plant and model outputs differ only slightly and the difference is reduced to zero quite rapidly. A detailed examination of convergence time is not essential in this situation. However, when a system is started with plant states greatly different from model states, it is important to know how design parameters can be selected to reduce this initial error to zero in a specified time. A closely related problem receiving attention in the recent literature[5,6] is that of the time-optimal and quasi time-optimal control problem.

In general, it is not an easy matter to choose parameters so that convergence time is minimized. In this report, the problem is presented of the design of an nth order plant such that convergence time is minimized. The analytical difficulties for plants higher than second order are so great that it was decided a more fruitful approach would be to arrive at results by intuition gained from the solution of the second order problem. The validity of these results for a third order plant was then checked using an analogue computer simulation. The computational part of this work was carried out in the Computer Center of the University of Connecticut which is supported in part by Grant No. GP-1819 of the National Science Foundation. The author is indebted to Mr. Donald Jorgenson for programming the IBM 7040 computer to solve the examples presented.

SYNTHESIS TECHNIQUE

The controller synthesis technique presented herein applies to single output plants (Fig. 1) which can be described by a set of n first order differential equations of the form

$$\dot{x}_i = x_{i+1} \qquad i = 1, 2 \,.....\, n-1 \qquad (1a)$$

$$\dot{x}_n = f(x_1, x_2, ...x_n, u, \dot{u}, ..u^m, r, \dot{r}, ...r^m, t) \qquad (1b)$$

or equivalently

$$\dot{\underline{x}} = \underline{f}\,(\underline{x}, \underline{u}, \underline{r}, t) \qquad (1c)$$

where r is the command input, u the control signal, and the x_i's the state variables.

The presence of derivatives of r and u in the function f allows for the possibility of plant zeroes.

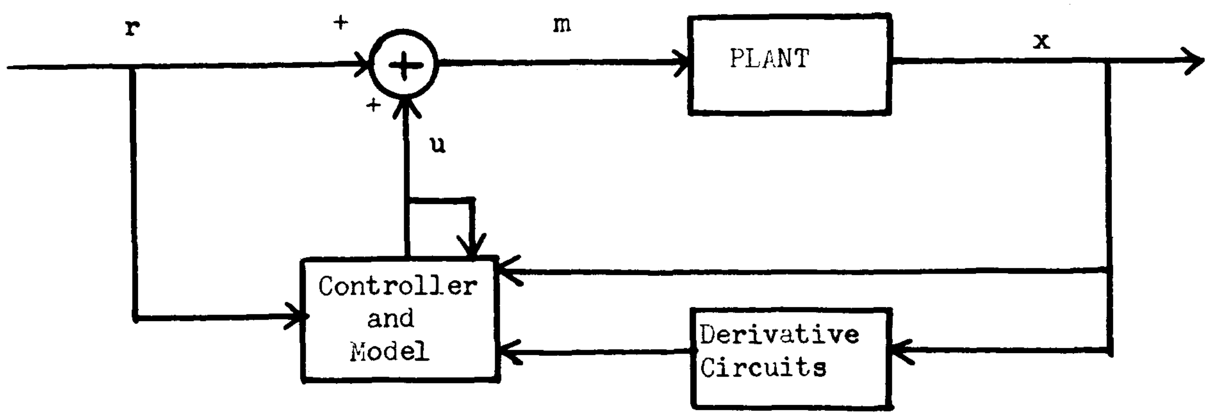


Fig. 1 - General System Configuration

With certain restrictions on the form of f which will be considered later, form (1) is necessary and sufficient for deriving a control law from a Liapunov function of quadratic form. This control law prescribes what control signal u is required to cause the behavior of the plant to be like that of a model reference with behavior governed by the linear constant coefficient vector differential equation

$$\dot{\underline{x}}_d = A_o \underline{x}_d + B_o \underline{r} \tag{2}$$

where $\underline{x}_d$ is an n vector with model state variables as elements, B_o is an nxn constant matrix, and A_o is a stable nxn constant matrix of the canonical form

$$A_o = \begin{bmatrix} 0 & 1 & 0 & 0 & . & 0 \\ 0 & 0 & 1 & 0 & . & \\ 0 & 0 & 0 & 1 & & \\ . & & & & & \\ a_{o1} & a_{o2} & a_{o3} & . & . & a_{on} \end{bmatrix} \tag{2a}$$

The form of (1) or (2a) implies either that the set of state variables chosen is comprised of the output signal and its first n-1 derivatives, or that an appropriate transformation from a different set of variables was used. It is interesting to note that the existence of such a transformation is a necessary and sufficient condition for a linear autonomous plant to be controllable[7,8].

In order to use the second method of Liapunov in controller design, it is necessary to have a differential equation with a well defined equilibrium, since the second method concerns itself with the stability of the equilibrium points of differential equations. In this design, it is through use of the model reference that an equation with a suitable equilibrium can be derived. This is done by defining an error vector

$$\underline{e} = \underline{x}_d - \underline{x} \tag{3}$$

The function of the controller is to cause $\underline{e}$ to approach zero. Since $\dot{\underline{e}} = \dot{\underline{x}}_d - \dot{\underline{x}}$, then (1c) can be subtracted from (2) to give a vector differential equation in the error variable

$$\dot{\underline{e}} = A_o\underline{e} + A_o\underline{x} - \underline{f}\ (\underline{x}, \underline{u}, \underline{r}, t) + B_o\underline{r} \tag{4}$$

It is to this equation that the second method is applied. The control effort is directed toward maintaining the equilibrium of (4), $\underline{e} = o$, asymptotically stable in the whole.

A function of the error states of quadratic form provides a convenient starting point for design. Let this function be

$$V(\underline{e}) = \underline{e}^T\ P\ \underline{e} \tag{5}$$

where P is a matrix to be determined. The time derivative of (5) is

$$\dot{V}(\underline{e}) = \underline{e}^T\ (A_o^T P + P\ A_o)\ \underline{e} + 2\underline{e}^T P \left[A_o\underline{x} - \underline{f}\ (\underline{x},\underline{u},\underline{r},t) + B_o\underline{r} \right] \tag{6}$$

The equilibrium, $\underline{e} = 0$, can be made asymptotically stable (and consequently (5) will be a Liapunov function for (4)) if

1. The solution of the equation $A_o^T P + P\ A_o = -Q$ (where Q is positive definite) yields a positive definite P.

2. $\underline{u}$ can be chosen to make the second term on the right hand side of (6) non-positive.

The first condition will be met as a consequence of choosing A_o to be a stable matrix. The second condition can be met provided the function f meets certain requirements, and its form and bounds are known.

Restrictions on the Function f

In the special case of a linear time-varying plant, the function f reduces to a linear combination of the plant states plus terms involving the elements of $\underline{u}$ and $\underline{r}$. For example

$$f = \underline{a}^T\underline{x} + \underline{b}^T\underline{u} + \underline{c}^T\underline{r}$$

where $\underline{a}$, $\underline{b}$, and $\underline{c}$ are n vectors with time-varying elements

$$\underline{a} = \begin{bmatrix} a_1(t) \\ a_2(t) \\ \cdot \\ \cdot \\ a_n(t) \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \cdot \\ \cdot \\ b_n(t) \end{bmatrix} \qquad \underline{c} = \begin{bmatrix} c_1(t) \\ c_2(t) \\ \cdot \\ \cdot \\ c_n(t) \end{bmatrix}$$

When f is of this form, the only condition it must satisfy is that the coefficient of the highest derivative of u be of one sign and non vanishing.

Example 1. - To illustrate the condition just described, consider the plant shown in Fig. 2. The equation describing this plant is

$$\ddot{x} + a\dot{x} + K_o x = K_o x + KT\dot{u} + Ku \tag{7}$$

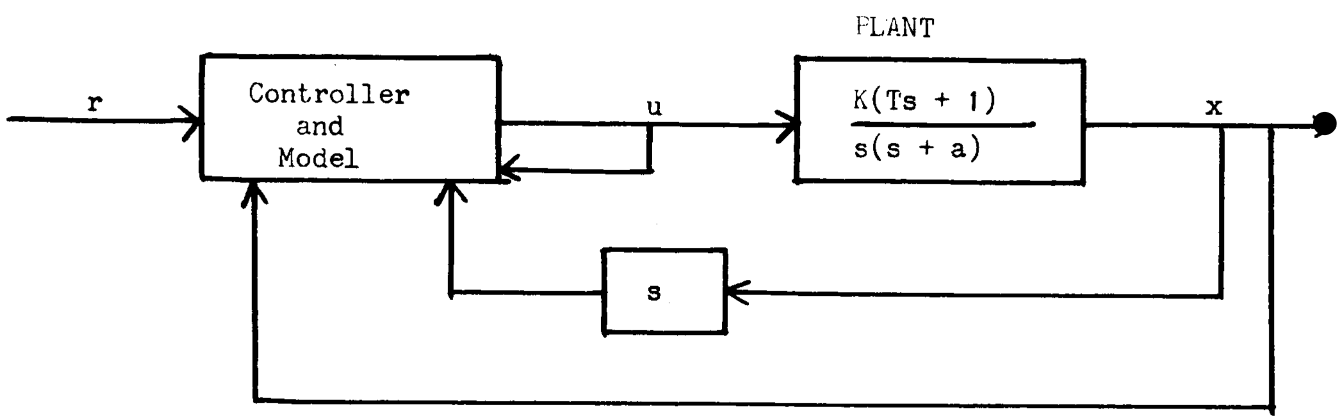


Fig. 2 - Second Order Plant With Zero

The desired behavior is given by the model equation

$$\ddot{x}_d + a_o\dot{x}_d + K_o x_d = K_o r \tag{8}$$

The error equation resulting from subtracting (7) from (8) is

$$\ddot{e} + a_o\dot{e} + K_o e = K_o(r-x) - KT\dot{u} - Ku + \alpha\dot{x} \tag{9}$$

where $\alpha = a - a_o$

for $V(\underline{e}) = \underline{e}^T P \, \underline{e}$,

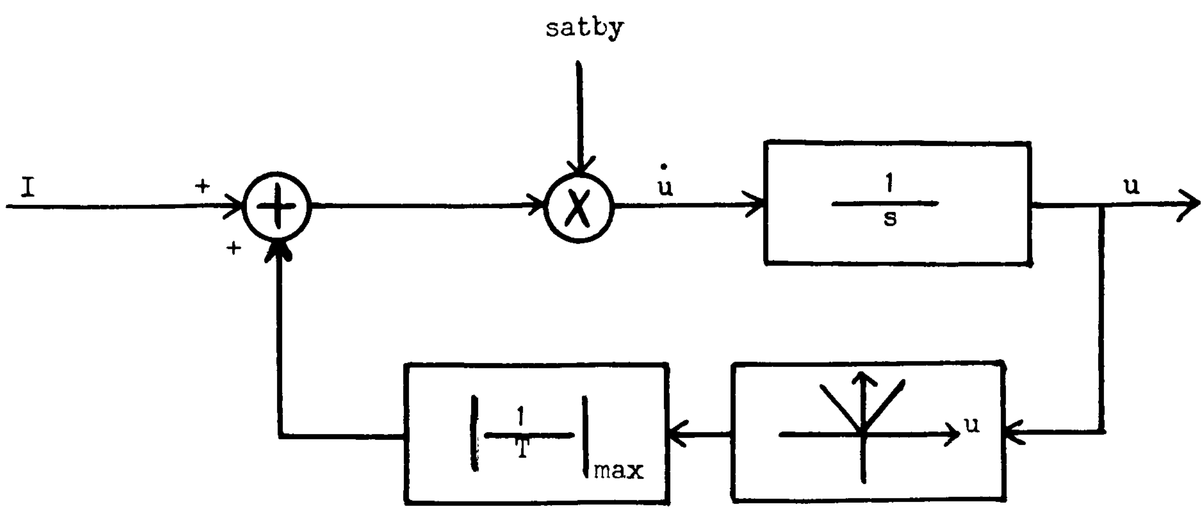
$$\dot{V} = -\left[q_{11}e_1^{\,2} + q_{22}e_2^{\,2}\right] - 2KT\,y\left[\dot{u} + \frac{1}{T}u - \frac{\alpha}{KT}\dot{x} - \frac{K_o}{KT}(r-x)\right] \tag{10}$$

where q_{11} and q_{22} are elements of a positive definite diagonal Q matrix, $y = p_{12}e_1 + p_{22}e_2$, and p_{12} and p_{22} are elements of the P matrix.

If $KT > 0$, i.e. if f satisfies the necessary condition for this linear time-varying plant, then a stabilizing control signal can be generated by choosing

$$\dot{u} = \left\{\left|\frac{1}{T}\right|^{\bullet}_{max}|u| + \left|\frac{\alpha}{KT}\right|^{\bullet}_{max}|\dot{x}| + \left|\frac{K_o}{KT}\right|^{\bullet}_{max}|r-x|\right\} \text{sign } y \tag{11}$$

The control signal u is generated as shown in the block diagram of Fig. 3. The saturation function is used rather than the sign function for reasons discussed in Ref. 4.



$$I = \left|\frac{\alpha}{KT}\right|^{\bullet}_{max}|\dot{x}| + \left|\frac{K_o}{KT}\right|^{\bullet}_{max}|(r-x)|$$

$$y = p_{12}\,e + p_{22}\,\dot{e}$$

Fig. 3. - Instrumentation For Generating The Control Signal

Note that K_0x has been added to both sides of (7). This is in contrast to the procedure described in Ref. 2 where use is made of unity linear feedback to obtain a term in x on the left hand side of the equation. In plants which contain at least one integrator, there is no need for the physical presence of this feedback, since the mathematical formulation given by (7) leads to a satisfactory control law for generating u. Leaving out the unity linear feedback path simplifies computation of the bounds of the coefficients of (7).

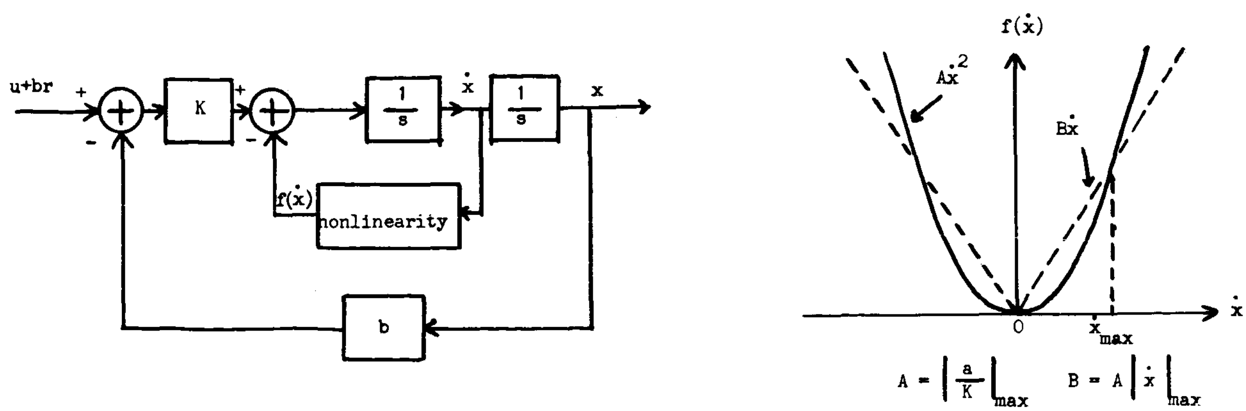
Another departure from previously reported procedure is that the reference input r is not fed directly into the plant. A plant with a zero was deliberately chosen to illustrate the advantage of this modification. If r were a direct input to the plant, then one of the terms in (11) would be $\dot{r}$. For an r characterized by step changes, this leads to the undesirable situation of requiring impulses in the generation of $\dot{u}$.

Previously, it has not been pointed out in the literature that maximum values of the coefficients of the variables in (11) should be chosen keeping in mind that if numerator, N(t), and denominator, D(t), time variations are simultaneously known, a reduction in coefficient magnitude may be achieved by using $|N(t)/D(t)|_{max}$, <u>not</u> $|N(t)|_{max}/|D(t)|_{min}$.
This oversight in Ref. 2 led to use of a coefficient larger than necessary by a factor of three, and thereby to an excessive control signal level.

The condition imposed on the function f in the case of nonlinear plants depends on its argument. If f is not a function of u or any of its derivatives, then the necessary condition for f to satisfy is exactly that just discussed for a linear time varying plant.

<u>Example 2</u>. - Consider the second order plant in Fig. 4a with square law damping. The equation for such a plant is

$$\ddot{x} + a\dot{x}^2 + Kbx = Ku + Kbr \tag{12}$$



(a) Plant (b) Nonlinearity

Fig. 4 - Second Order Plant With Square Law Damping

The model and Liapunov function used in Example 1 are used here and in all further examples. $\dot{V}$ in this case is

$$\dot{V} = - \left[q_{11}e_1^{\;2} + q_{12}e_2^{\;2} \right] - 2\,Ky \left[u + \frac{Kb-K_o}{K}(r-x) + \frac{a_o\dot{x}}{K} - \frac{a\dot{x}^2}{K} \right] \tag{13}$$

For b = 1, the control law required to keep $\dot{V}$ negative is

$$u = \left\{ \left|\frac{K-K_o}{K}\right|_{max} \left|(r-x)\right| + \left|\frac{a_o}{K}\right|_{max} \left|\dot{x}\right| + \left|\frac{a}{K}\right|_{max} \dot{x}^2 \right\} \text{sign } y \tag{14}$$

In (13), $\dot{x}^2$ can be instrumented with a multiplier. However, if an upper bound, $|\dot{x}|_{max}$, on $|\dot{x}|$ can be determined, the multiplier is unnecessary and a simple gain for $|\dot{x}|$ will suffice. This is illustrated in Fig. 4b.

$$\text{Since} \quad |a/K|_{max} |\dot{x}|_{max} > |a/K|_{max} \dot{x}^2 \quad \text{for } |\dot{x}| < |\dot{x}|_{max}, \tag{15}$$

then the term $|a/K|_{max}\, \dot{x}^2$ in (14) can be replaced by $|a/K|_{max} |\dot{x}|_{max} |\dot{x}|$ if operation is restricted to the range where $|\dot{x}| < |\dot{x}|_{max}$.

Since the nonlinearity of this example is not a function of u, the only condition which f must satisfy is that K > 0. This is evident from (13) and (14).

Example 3. - Plants which exhibit either static friction or coulomb friction can be controlled using this technique (Fig. 5).

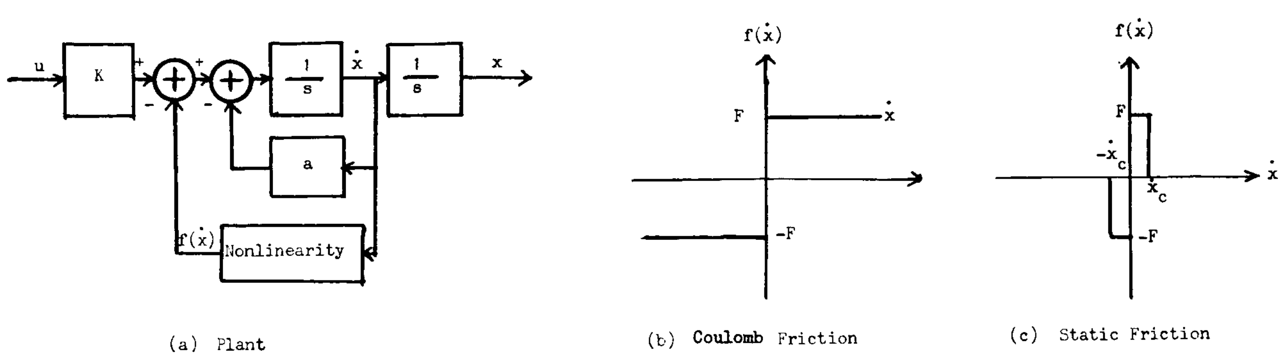


Fig. 5 - Second Order Plant With Static or Coulomb Friction

The equation describing the behavior of these plants is

$$\ddot{x} + a\dot{x} + f(\dot{x}) = Ku \tag{16}$$

Application of the design equations yields

$$u = \left\{ \left|\frac{K_o}{K}\right|_{max} |r-x| + \left|\frac{\alpha}{K}\right|_{max} |\dot{x}| + \left|\frac{1}{K}\right|_{max} F \right\} \text{sign } y \qquad (17)$$

The control signal given by (17) is adequate for either the static friction or the coulomb friction case. However, in the case of static friction, the constant term involving F is not necessary for $|\dot{x}| > |\dot{x}_c|$. This fact can be used to advantage to reduce the magnitude of u by including a relay in the design which removes the constant signal from u when $|\dot{x}| > |\dot{x}_c|$.

Example 4. - Another example illustrating the design technique for nonlinear plants is the plant of Fig. 6 for which the equation is

$$\ddot{x} + a\dot{x} + bx + cx^3 = Ku + K \int_0^t u\,dt \qquad (18)$$

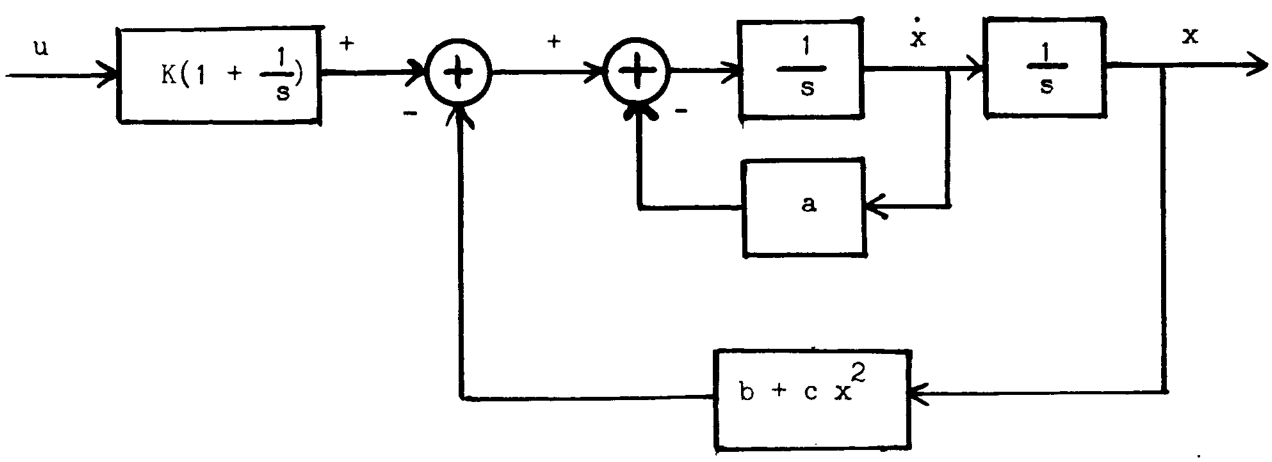


Fig. 6 - Second Order Plant With Hard and Soft Spring Type Nonlinearity

In (18), $c > 0$, $c = 0$, and $c < 0$, corresponds to a hard spring, linear spring, and soft spring respectively. The design procedure is straightforward and leads to the control signal

$$u = \left\{ \left|\frac{K_o}{K}\right|_{max} \cdot |(r-x)| + \left|\frac{\alpha}{K}\right|_{max} \cdot |\dot{x}| + \left|\frac{b}{K}\right|_{max} \cdot |x| + \left|\frac{c}{K}\right|_{max} \cdot |x^3| + \left|\int_0^t u\,dt\right| \right\} \text{sign } y \qquad (19)$$

where $\alpha = a - a_o$.

The modified forcing function on the right hand side of (18) in Example 4 is necessary because the plant has no pure integrator. Consider r to be a step input, R, to the model. In this case means must be found to provide a steady state input to the plant. Since in normal operation of the controller, $u \to 0$ as $\underline{e} \to 0$, the control signal itself cannot perform this function. However, the integral of this signal, appearing on the right hand side of (18), can. In the steady state, $\underline{e} = 0$, $x = R$ and $K \int_o^t u dt = bR + cR^3$.

The last form of f which will be considered is one which is a function of u. This case, which arises in a plant with a nonlinear gain in the forward path, has been treated previously in Ref. 4. The condition required of f is shown there to be

$$\infty > \frac{f(u)}{u} > 0$$

A nonlinear gain is a common form of plant nonlinearity. As shown in Ref. 4, this type of nonlinearity is also handled by a straightforward application of this technique, and the resulting controllers are easily implemented.

Computer Simulation of Several Examples

Several cases each of Examples 2, 3, and 4 were simulated on the IBM 7040 digital computer. The results of these simulations are shown in Figures 7 through 13. The model used in all cases is that given by (8) with $a_o = K_o = 2$. The input to the system, r(t), is taken as a unit step function in some cases and as a sine or cosine of unit amplitude and frequency $\omega = 0.1$ in others. The percentage error is indicated at several points on each of the error curves. This percentage is $((x_d - x)/x_d)\cdot(100)$. The conditions which pertain for each of the Figures 7 through 13 are described in Table I. In all cases, the control signal was generated using sat $10(2e + 3\dot{e})$. The error can be made smaller by using a $b > 10$, but even for $b = 10$, it is quite small most of the time.

CONVERGENCE TIME

The control signal generated by the controller guarantees that plant states approach the model states. To obtain an estimate of convergence time a parameter η is defined as

$$\eta = \min \left[\frac{-\dot{V}(e,t)}{V(\underline{e})} \right], \quad \underline{e} \neq 0 \tag{21}$$

TABLE I

Conditions for Figs. 7 - 13

Fig.	Example	Case	Plant Equation	Coefficients	Input, r(t)
7	2	1	(12)	a = 1 b = 0 K = 1	1
8	2	2	(12)	a = sin t b = 1 K = 1	1
9	2	3	(12)	a = sin t b = 1 K = 1	sin 0.1 t
10	3	1	(16) $f(\dot{x})$ of Fig. 5c F=1, $\dot{x}_c$ = 0.1	a = sin t K = 1	cos 0.1 t
11	3	2	(16) $f(\dot{x})$ of Fig. 5b F = 1	a = sin t K = 1	sin 0.1 t
12	4	1	(18)	a = sin t b = 1 c = 1 K = 1	1
13	4	2	(18)	a = sin t b = 1 c = 1 K = 1	cos 0.1 t

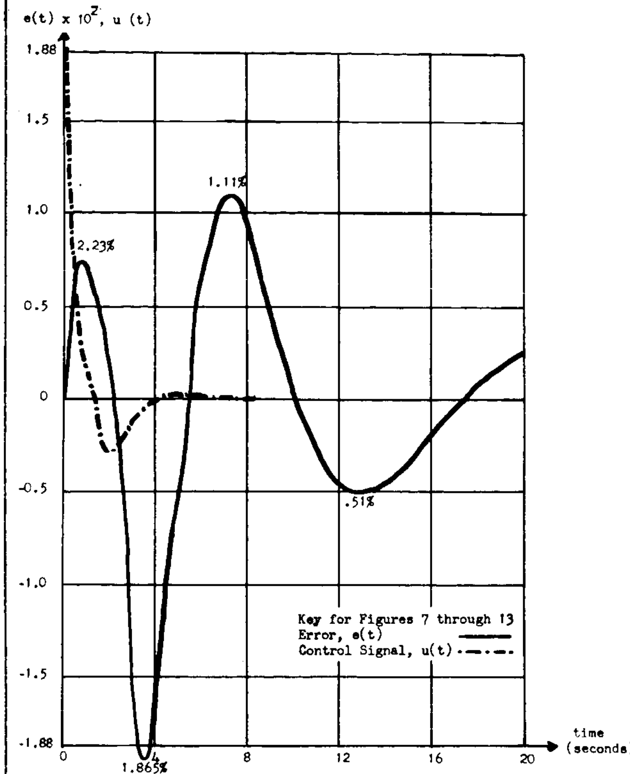


Fig. 7 - e(t) and u(t), Ex. 2, Case 1

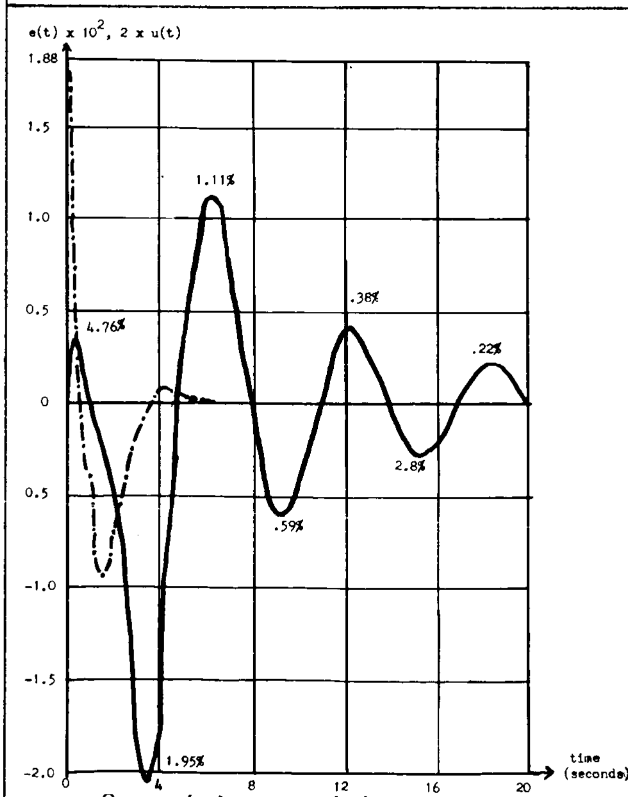


Fig. 8 - e(t) and u(t), Ex. 2, Case 2

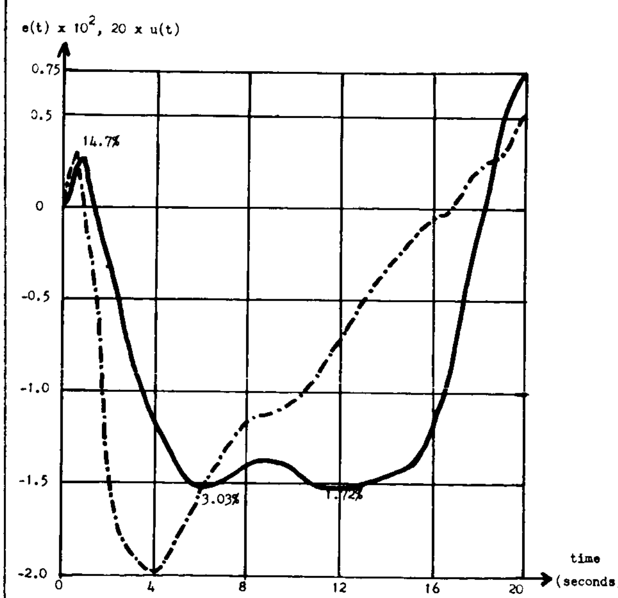


Fig. 9 - e(t) and u(t), Ex. 2, Case 3

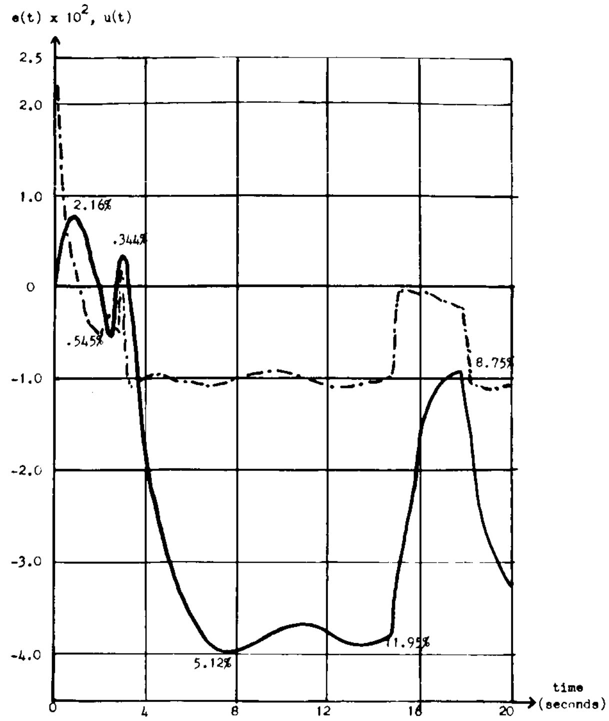


Fig. 10 - e(t) and u(t), Ex. 3, Case 1.

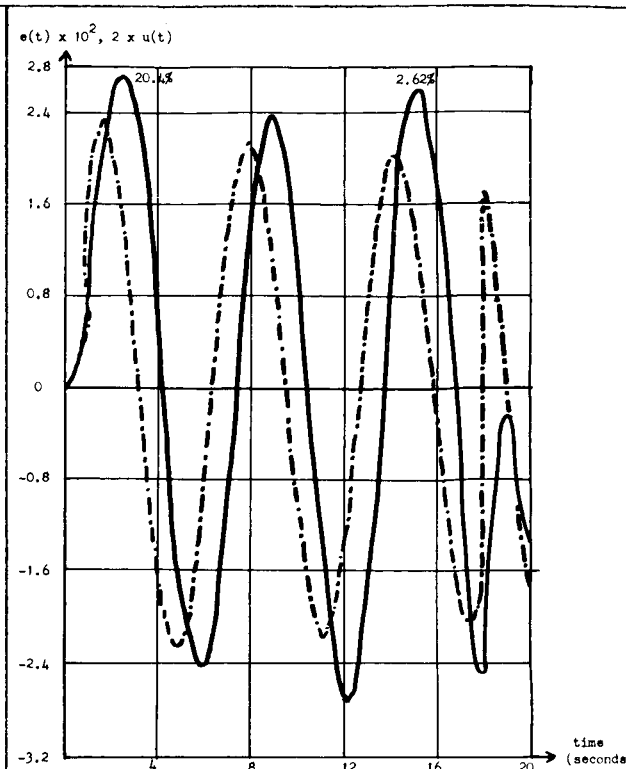


Fig. 11 - e(t) and u(t), Ex. 3, Case 2.

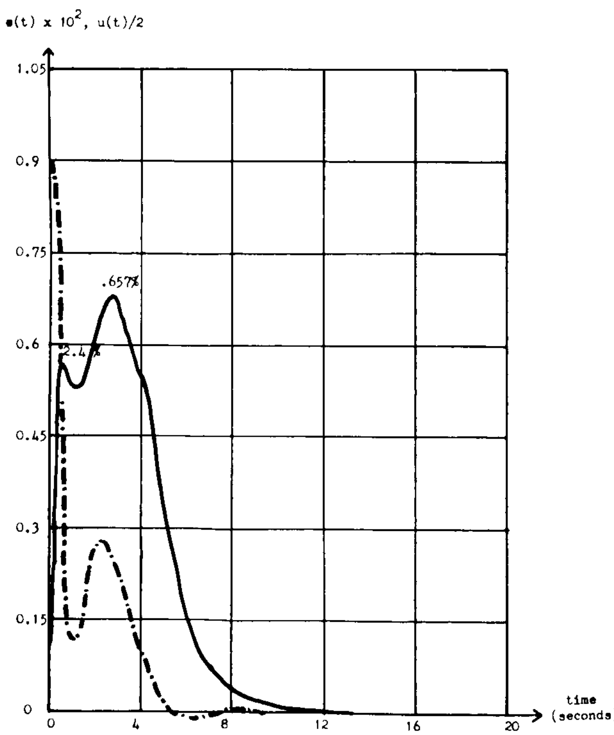


Fig. 12 - e(t) and u(t), Ex. 4, Case 1.

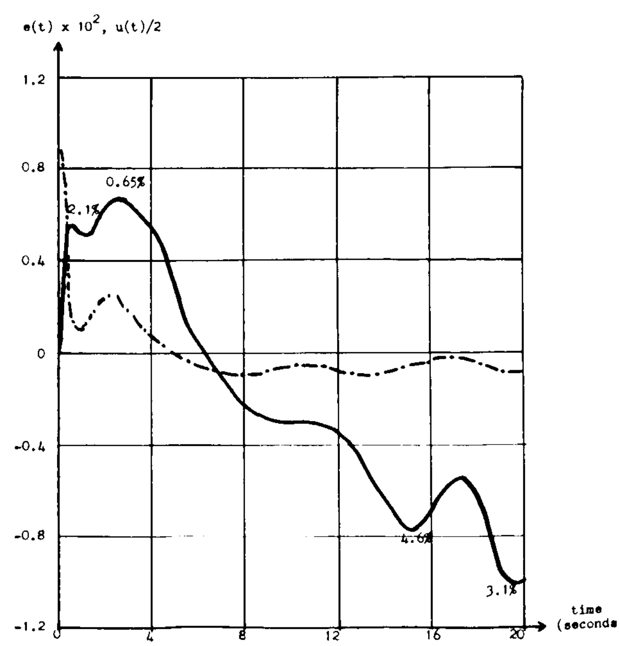


Fig. 13 - e(t) and u(t), Ex. 4, Case 2.

From (21)

$$\dot{V}(\underline{e},t) \leq -\eta V(e) \tag{22}$$

This last equation can be solved to yield

$$V(\underline{e}) \leq V(\underline{e}_o) e^{-\eta\ (t-t_o)} \tag{23}$$

where

$$V(e_o) = V(\underline{e},\ t_o) \tag{24}$$

It is seen from (23) that the design should be directed toward maximizing the value of η if convergence time is to be minimized.

Since $\dot{V}$ depends on u, a nonlinear, non algebraic function, the actual value of η is difficult to compute. This difficulty is avoided by defining the two additional quantities,

$$\dot{V}_o(\underline{e}) \triangleq -\underline{e}^T Q\ \underline{e} \geq \dot{V}(\underline{e},t) \tag{25}$$

and

$$\eta_o \triangleq \frac{-\dot{V}_o(\underline{e})}{V(\underline{e})} \leq \eta \tag{26}$$

Since η_o is the ratio of two algebraic functions of e, e_1 .... e_n , the problem of maximizing it is easier to handle than that of maximizing η .

If (26) is written as

$$\eta_o = \frac{-\underline{e}^T(A_o^T P + PA_o)\underline{e}}{\underline{e}^T P\ \underline{e}} = \frac{\underline{e}^T\ Q\underline{e}}{\underline{e}^T\ P\underline{e}} \tag{27}$$

then it is clear that the convergence time depends explicitly on the P matrix of the Liapunov function and the A_o matrix of the model. Since the design procedure involves arbitrarily choosing a positive definite matrix Q, then P is determined once the choices for Q and A_o are made. The design problem then is to choose Q and A_o so as to minimize convergence time.

A conservative estimate of convergence time can be obtained by use of the fact that a positive definite quadratic form is bounded above and below by the inequality

$$\min_i \lambda_i\ (P)\ ||\underline{e}||^2 \leq \underline{e}^T\ P\ \underline{e} \leq \max_i \lambda_i\ (P) ||\underline{e}||^2 \tag{28}$$

where λ_i (P) are the eigenvalues of P for i = 1, 2, ...n. These eigenvalues are positive since P is a positive definite matrix. $||\underline{e}||^2$ symbolizes the square of the Euclidian norm of $\underline{e}$. Use of (28) allows the simple estimate for η,

$$\eta \geq \eta_o \geq \frac{\min\limits_i \lambda_i (Q)}{\max\limits_i \lambda_i (P)} = b > 0 \tag{29}$$

Although (29) allows a simple estimate for η once Q and A_o are chosen, it does not give an explicit relation from which design parameters can be chosen to yield a specified value of η. In addition, the estimate afforded by (29) is too conservative to have much meaning in the physical problem. To acquire further insight into the nature of the parameter η, consideration is directed toward the second order problem.

Relation of Convergence Time to Design Parameters for the Second Order Case

To determine how η_o is affected by the design parameters in the second order case, it is written in terms of these parameters as

$$\eta_o = \frac{q_{11}e^2 + q_{22}\dot{e}^2}{p_{11}e^2 + 2p_{12}e\dot{e} + p_{22}\dot{e}^2} \tag{30}$$

where a diagonal Q matrix has been assumed.

Solving $A_o^T P + PA_o = - Q$ for P yields

$$p_{11} = \frac{K_o}{2a_o}\left(\frac{q_{11}}{K_o} + q_{22}\right) + \frac{a_o q_{11}}{2K_o} \tag{31a}$$

$$p_{12} = \frac{q_{11}}{2K_o} \tag{31b}$$

$$p_{22} = \frac{q_{11}}{2K_o a_o} + \frac{q_{22}}{2a_o} \tag{31c}$$

where the A_o for the model given by (8) has been used.

These expressions for the P matrix elements are substituted into (30) to give

$$\eta_o = \frac{e^2 + \dot{e}^2}{\left[\frac{1 + \beta K_o}{2a_o} + \frac{a_o}{2K_o}\right] e^2 + \frac{1}{K_o} e\dot{e} + \left[\frac{1 + \beta K_o}{2a_oK_o}\right]\dot{e}^2} \tag{32}$$

where

$$\beta = \frac{q_{22}}{q_{11}}$$

The parameter β indicates the relative weighting of $\dot{e}^2$ and e^2 in $\dot{V}_o(e)$.

To find the minimum value of η_o, its derivatives with respect to e and $\dot{e}$ are taken in (32) and set equal to zero. In both cases the equation to be satisfied is

$$e^2 + \left[\frac{1}{a_o} - \frac{(\beta K_o)^2}{a_o} - \beta a_o\right] e\dot{e} - \beta \dot{e}^2 = 0 \tag{33}$$

The solution of (33) yields

$$e = k_1 \dot{e} \tag{34}$$

where

$$k_1 = -\frac{1}{2}\left[\frac{1-(\beta K_o)^2}{a_o} - \beta a_o\right] \pm \frac{1}{2}\sqrt{\left[\frac{1-(\beta K_o)^2}{a_o} - \beta a_o\right]^2 + 4\beta}$$

If (34) is used in (32) the result is

$$\eta_o = \frac{(k_1^2 + \beta)(2a_o K_o)}{(\beta K_o{}^2 + K_o + a_o{}^2)\, k_1^2 + 2a_o k_1 + 1 + \beta K_o} \tag{35}$$

All of the parameters to be chosen in design, i.e. a_o, K_o, and β, are brought out explicitly in (35).

Since η_o is a function of the three parameters a_o, K_o, and β, it is not in general an easy task to maximize its minimum value by choice of these parameters. A significant simplification of (34) and (35) occurs if β is chosen such that

$$\beta K_o << 1 \tag{36}$$

This leads to

$$k_1 \approx -\frac{1}{a_o} \qquad (37a) \qquad \text{and} \qquad k_1 \approx +\beta\, a_o \qquad (37b)$$

The value for k_1 given by (37b) gives the minimum value for η_o which is

$$\eta_o = \frac{(1+\beta a_o^{\,2})\;\; 2a_o \beta\, K_o}{(1+\beta a_o^{\,2})^2 + \beta\, K_o + \beta^2 a_o^{\,2}\, K_o\, (1+\beta\, K_o)} \qquad (38)$$

For the second order system given by (8)

$$a_o = 2\,\zeta\sqrt{K_o} \qquad (39)$$

where ζ is the damping ratio and $\sqrt{K_o}$ the undamped natural frequency of the model.

Use of (39) and (38) leads to

$$\eta_o = \frac{(1+4\zeta^2\beta\;K_o)\;4\zeta\sqrt{K_o}\,\beta\,K_o}{(1+4\zeta^2\beta\;K_o)^2 + \beta\,K_o + 4\zeta^2\beta^2 K_o^{\,2}\,(1+\beta\,K_o)} \qquad (40)$$

From (40) it is seen that for a fixed damping ratio , the convergence time can be made as small as desired by increasing K_o while keeping $\beta\;K_o$ constant at a value which satisfies (36).

Consider the significance of the condition (36) if the model parameters are fixed. For $K_o > 1$, $\beta << 1$. This means that the error is weighed much more heavily in the $\dot{V}$ function than its derivative. This weighting can be interpreted in terms of the switching line defined by the switching function of (11). The equation for this line is

$$\dot{e} = -\frac{p_{12}}{p_{22}}\; e = -\frac{a_o}{1+\beta\,K_o}\; e \qquad (41)$$

In (41), if $\beta\;K_o << 1$, then

$$\dot{e} \approx -a_o\, e \qquad (42)$$

and the slope of this switching line has its maximum possible magnitude. Thus, the effect of the weighting of states in the Q matrix has been to rotate the switching line toward the $\dot{e}$ axis in the e-$\dot{e}$ plane. The effect of this rotation in decreasing convergence time is illustrated in Fig. 14.

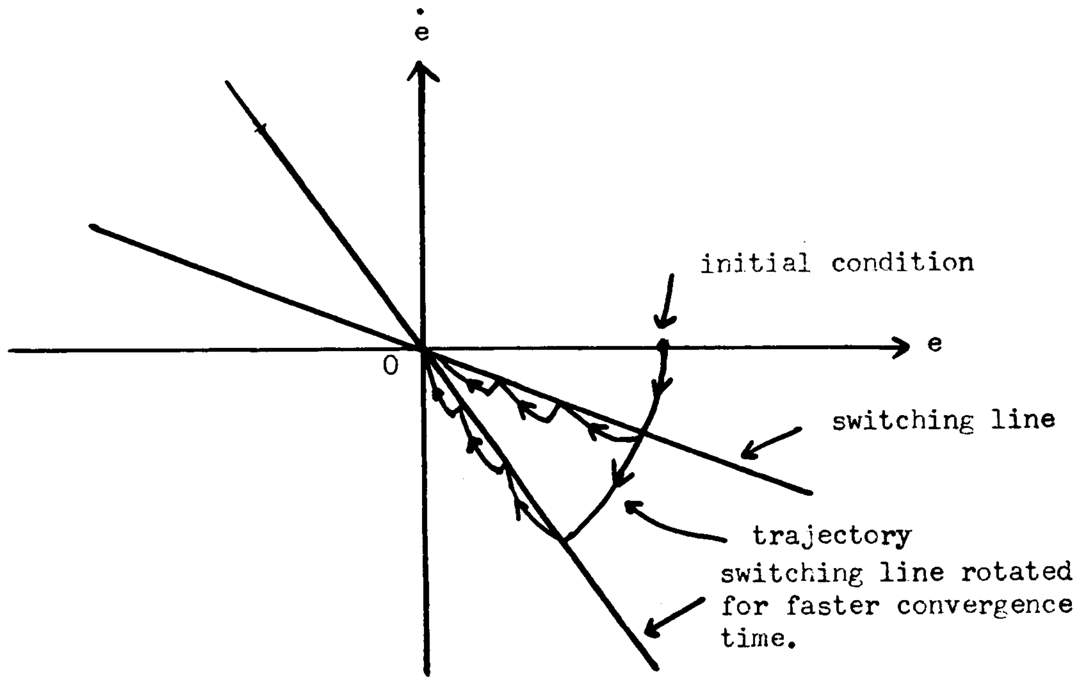


Fig. 14 - Relation of Switching Line Slope to Convergence Time

Higher Order Systems

An analysis similar to that above for the second order problem would be extremely difficult to carry out for even a third order system. Rather than using an exact analysis, the results of the second order case were applied to an analogue computer simulation of a third order plant. The transient response for the error variable was compared for two different Q matrices. The plant used is described by the equation

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & 0 \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \\ \ddot{u} \end{bmatrix} \tag{43}$$

and the model by

$$\dot{\underline{x}}_d = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.2 & -2.15 & -1.3 \end{bmatrix} \underline{x}_d + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1.2 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ \dot{r} \\ \ddot{r} \end{bmatrix} \tag{44}$$

The plant described by (43) represents the pitch axis stability augmentation system of the X-15 manned re-entry vehicle[9].

To carry over the results of the second order case, the control law was derived first with a Q matrix equal to the identity matrix (the one usually chosen for convenience in the literature), and then with a Q matrix in which the error variable was weighed more heavily than its derivatives.

These matrices are

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.01 \end{bmatrix} \tag{45}$$

The switching function part of the control law resulting from each of these is

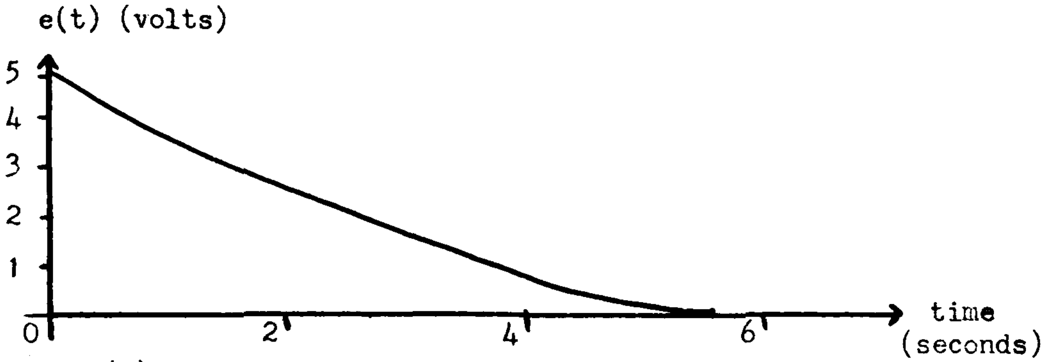
$$y_1 = 0.415e + 1.23\dot{e} + 1.33\ddot{e} \tag{46a}$$

and

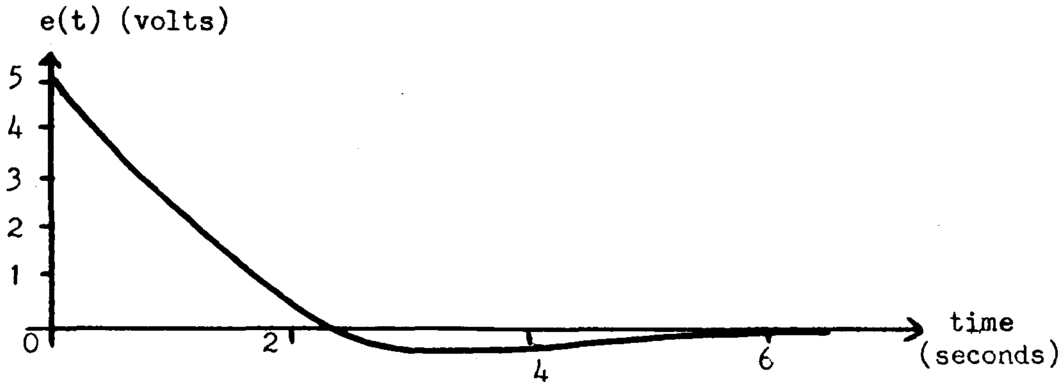
$$y_2 = 0.415e + 0.486\dot{e} + 0.377\ddot{e} \tag{46b}$$

where y_1 corresponds to Q_1 and y_2 to Q_2.

The system was started with initial conditions $e = 5$, $\dot{e} = \ddot{e} = 0$. The transient response for both cases is shown in Fig. 15. It is seen that the one for Q_2 is considerably faster than that for Q_1.



(a) Transient for Q_1



(b) Transient for Q_2

Fig. 15 - Transient Response For Third Order System

CONCLUSIONS

Controllers can be designed for a wide class of nonlinear plants using the technique presented in this report. The technique is applicable to plants for which the form of the nonlinearity is known. Digital computer results presented for plants with square law damping, static friction, coulomb friction, and hard and soft spring type nonlinearities show that the control effects close agreement between the plant and model reference outputs.

An exact expression for convergence time in terms of design parameters is derived for the second order case. Derivation of an equivalent expression for higher order systems is quite complex. However, the solution of the second order problem does lead to insight which is useful in designing higher order systems.

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